O404. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$(a+b+c)^2\left(\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\right)\geq 9$$

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Since by AM-GM Inequality

$$ab+bc+ca \geq 3\sqrt[3]{a^2b^2c^2} = 3$$

then

$$(a+b+c)^2 = a^2+b^2+c^2+2\left(ab+bc+ca\right) \geq a^2+b^2+c^2+6 = \sum_{cyc}\left(a^2+2\right)$$

and by Cauchy-Schwarz inequality

$$\sum_{cyc}\left(a^2+2\right)\cdot\sum_{cyc}\frac{1}{a^2+2} \geq 9\ .$$

Therefore,

$$(a+b+c)^2\sum_{cyc}\frac{1}{a^2+2} \geq \sum_{cyc}\left(a^2+2\right)\cdot\sum_{cyc}\frac{1}{a^2+2} \geq 9.$$

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